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# Minimax optical flow estimation from a sequence of 2D images

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## 1 Introduction

Apparent motion estimation, from an image sequence, is one crucial issue for the Image Processing community in a wide range of applicative domains. When dealing with environmental applications, it allows to study the tracking of clouds on satellite meteorological data and the surface circulation on ocean acquisitions. During the last twenty years, many authors investigated the issue of fluid flow motion estimation, see for instance [4] for a detailed survey.

Motion estimation is an ill-posed problem, according to Hadamard definition [3], as the solution retrieved from the image data is not unique. This ill-posedness comes from the fact that the equations, used to model apparent motion, are under-determined: this is the well-known aperture problem. An additional constraint is required to compute a unique velocity field from image acquisitions. A usual way is to restrict the dimension of the space of admissible solutions. For instance, the result may be searched among the

functions with bounded spatial variations, which is called “Tikhonov regularization method” in the literature [7].

If the temporal period of image acquisitions is large compared to the underlying dynamic processes of the observed system, these image data are only snapshots of the observed brightness function. In that case, an improved motion estimation is obtained by using a dynamic model of the motion field, in place of applying a Tikhonov regularization. In such context, application of data assimilation methods emerged around five-six years ago. Readers can refer to [6], [8], and [1] for recent contributions to the subject, with various mathematical approaches.

However, the image data are uncertain due to various errors occurring during the acquisition process. The dynamic model is also uncertain and only approximates the processes, that are underlying the image evolution. Estimating motion is then not sufficient. An uncertainty measure is required in order to exploit results. A solution to define an uncertainty measure for the estimate is to use the minimax approach.

One way to overcome the dimension issue, that arises with minimax methods, is to exploit the dual structure of the optical flow constraint, as explained in this paper. On one hand, the optical flow constraint expresses how the brightness function is advected in time by the given motion field. So, given a space-time motion field and the initial condition, one may compute the brightness function. On the other hand, given the brightness function, one might estimate its spatio-temporal gradient and use the optical flow constraint as an algebraic equation. The latter equation together with the dynamical model, assumed for the motion field, allows one to compute the motion field. This duality principle is used, in the paper, in order to split the estimation procedure into two parts. In the first one, the motion field is fixed and a continuous estimate of the brightness function is obtained by the minimax method from the observed images with the optical flow constraint. This is a linear-quadratic problem, and the analytical form of the minimax estimate is available. In the second part of the estimation, we compute the minimax estimate of the brightness function’s gradient and plug it into the advection part of the optical flow constraint. We also plug the previous motion field, which was used to compute the minimax estimate of the brightness function, into the advection part of the Navier-Stokes equation. As a result, we obtain a system of two linear PDEs, that allows to determine the analytical form of the minimax estimate for the motion field from the observed images. That two-part estimation process is iterated until the convergence of the motion field. As a result, the minimax approach becomes applicable for the issue of optical flow estimation on large size image sequence.

The split of the estimation in two parts relies on the following remarks. The motion field defines the advection of brightness in time. If this is fixed, then it becomes possible to optimally fit a brightness function to the observed image snapshots. Afterwards, spatial gradient of this function may be computed. Next, the optical flow equation is available as observation equation in order to fit an improved motion field to the image snapshots.

Even if the two-part process relies on the optical flow constraint and Navier-Stokes equations, in the same way than data assimilation methods [1], the paper demonstrates that the approach provides major advantages. First the split in two parts allows to successively consider and solve two linear problems, and to construct, at each iteration, a minimax estimate for brightness function and motion field. These estimates are associated with an uncertainty measure, which is the first major improvement compared to data assimilation. As minimax do not rely to a model of the image and model uncertainties, there is no need to assume a Gaussian uncertainty of parameters, as this is the case in 4D-Var. This is the second advantage of the method. Last, the method allows to address rigorously the issue of sparse temporal data, as this is usually the case when dealing with satellite acquisitions. However, the major drawback of the approach is the requirement on an initial estimate of motion field, at the first iteration.

This paper is organized as follows. We provide all necessary notation in section 2. Problem statement is given in section 3. The algorithm is described in section 4. A brief conclusion is presented in section 5.

## 2 Notation

The section describes mathematical notation used in the paper.

Let  $\langle \cdot, \cdot \rangle$  denotes the canonical inner product in the abstract Hilbert space  $H$ , with norm  $\| \cdot \|_H$ :  $\|f\|_H^2 := \langle f, f \rangle$  for  $f \in H$ .

Let  $(t_0, T)$  be a bounded open subset of the real line, and define a space  $\mathbb{L}_2(t_0, T, H)$  of functions  $f : (t_0, T) \rightarrow H$  such that:

$$\|f\|_{\mathbb{L}_2}^2 := \int_{t_0}^T \|f(t)\|_H^2 dt < +\infty$$

$\Omega$  is a bounded open subset of  $\mathbb{R}^2$  with Lipschitz boundary  $\Gamma := \partial\Omega$ , and  $\Omega_T := \Omega \times (t_0, T)$ . Let define a Neumann operator  $N_\Gamma I := \nabla I \cdot \mathbf{n}$ , where  $\mathbf{n}$  denotes a normal vector pointing outside the domain  $\Omega$ .

Let  $L_2(\Omega)$  denote the space of functions  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$  such that

$$\|f\|_{L_2}^2 := \int_{\Omega} f^2(x, y) dx dy < +\infty$$

and let  $\mathbb{L}_2(\Omega) := L_2(\Omega)^2 = L_2(\Omega) \times L_2(\Omega)$ . Define by  $L_{\infty}(\Omega_T)$  the set of measurable functions bounded almost everywhere on  $\Omega_T$ . Let  $C^0(t_0, T, L_{\infty}(\mathbb{R}^2))$  denote the space of all continuous functions from  $(t_0, T)$  to  $L_{\infty}(\mathbb{R}^n)$ .

Denote by  $H^m(\Omega)$  a Sobolev space of functions  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$  with the norm:

$$\|f\|_m^2 := \sum_{\alpha_1 + \alpha_2 \leq m} \|\partial_x^{\alpha_1} \partial_y^{\alpha_2} f\|_{L_2}^2$$

where  $\partial_x$  denotes the weak derivative of the function  $f$  with respect to  $x$ . We define  $\mathbb{H}^m(\Omega) := H^m(\Omega)^2$ . For  $\mathbf{v} = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \in \mathbb{H}^m(\Omega)$  we define  $\text{curl } \mathbf{v}(x, y, t) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$  and  $\text{div } \mathbf{v} = \partial_x u + \partial_y v$ . Let us also set  $\nabla^{\perp} := (\partial_y, -\partial_x)^T$  and  $\Delta = \nabla^2 = \partial_x^2 + \partial_y^2$ . Let  $\mathbf{I}$  denote the identity matrix.

### 3 Problem statement

Assume that the brightness function  $I(x, y, t) : \Omega_T \rightarrow [0, 1]$  is observed at points  $\{(x_k, y_l)\}_{k, l=1}^{N_x, N_y}$  of the image domain  $\Omega$  and  $Y_s^{kl} \in [0, 1]$  represent the observed values  $I(x_k, y_l, t_s)$  in the following form:

$$Y_s^{kl} = \int_{\Omega_T} g_s^{kl}(x, y, t) I(x, y, t) dx dy dt + \eta_s^{kl}, \quad k = \overline{1, N_x}, l = \overline{1, N_y}, s = \overline{1, S}, \quad (1)$$

where  $g_s^{kl}$  encapsulates the acquisition procedure, and  $\eta_s^{kl}$  stands for the acquisition noise.

A point  $P = (x, y)$  of the domain  $\Omega$  is supposed transported by the velocity field  $\mathbf{v} := (u(x, y, t), v(x, y, t))^T$  and its brightness is approximately conserved in time along its trajectory:  $I(x, y, t) \approx \text{const}$ . In other words, the brightness function satisfies the so-called optical flow constraint [5]:

$$\frac{d}{dt} I(x, y, t) = \partial_t I + u(x, y, t) \partial_x I + v(x, y, t) \partial_y I = e^o(x, y, t) \quad (2)$$

where we assumed, for a moment, that all partial derivatives of  $I$  exist and  $e^o \in \mathbb{L}_2(t_0, T, \mathbb{L}_2(\Omega))$  models the uncertainty of this Lagrangian constancy hypothesis. One may also say that  $I$  is advected by the motion field  $\mathbf{v}$ .

Let us also assume that the motion field  $\mathbf{v} = \begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix}$  is a weak solution of the following 2D Navier-Stokes Equation (NSE):

$$\begin{aligned} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p &= \nu \Delta u + e_u^m, \\ \partial_t v + u\partial_x v + v\partial_y v + \partial_y p &= \nu \Delta v + e_v^m, \\ \partial_x u + \partial_y v &= 0, \quad (x, y, t) \in \Omega_T, \\ \mathbf{v}(x, y, t) &= 0, \quad (x, y) \in \partial\Omega, \\ \mathbf{v}(x, y, t_0) &= \mathbf{v}_0(x, y) + \mathbf{e}^b(x, y), \quad (x, y) \in \Omega, \end{aligned} \tag{3}$$

with  $\mathbf{v}_0$  standing for an initial motion field and  $e_u^m, e_v^m, \mathbf{e}^b$  describing model error and uncertainty in the initial condition.

Our aim is to construct the minimax estimate of the motion field  $\mathbf{v}$ , given discrete snapshots  $Y_s^{kl}$ , and assuming that the uncertain parameters  $e^o, e_u^m, e_v^m, \mathbf{e}^b$  belong to a given bounded convex set of  $L_2(\Omega_T)$  and that  $\eta_s^{kl}$  are independent scalar random variables with zero mean and covariances  $R_s^{kl}$ .

## 4 Iterative minimax optical flow estimation

In this section, we describe the algorithm, that has been summarized in section 1, and which allows to solve the problem described in section 3. In particular, it aims to: first, optimally resolve the issue of sparse temporal acquisitions; second, avoid modelling the uncertainty parameters by Gaussian variables; and last provide an uncertainty measure associated to the estimation. As explained in the introduction, the solution is based on a duality idea. On the one hand, given a motion field  $\mathbf{v}^* = (u^*, v^*)^T$  one constructs an estimate  $\hat{I}^*$  of the brightness function such that  $\hat{I}^*$  optimally fits the given snapshots  $Y_s^{kl}$  and solves the optical flow constraint (2) with  $u = u^*, v = v^*$ . In other words, for the given motion field  $\mathbf{v}^*$  we construct  $\hat{I}^*$ , determined by this motion field through the optical flow constraint (2), which a) optimally fits the observed data  $Y_s^{kl}$  and b) takes into account measurement errors  $\eta_s^{kl}$  together with possible uncertainty in the motion field  $\mathbf{v}^*$ . We stress that a) and b) are not satisfied if one generated the brightness function just advecting the given initial image with the motion field  $\mathbf{v}^*$ . On the other hand, given the estimate  $\hat{I}^*$  of the brightness function one gets the estimate  $\widehat{\nabla I}^*$  of its gradient. Then, plugging  $\mathbf{v}^*$  into advection part of (3) and substituting  $\nabla I$  with  $\widehat{\nabla I}^*$  in (2) one gets a system of linear equations for  $I$  and  $\mathbf{v}$  which is then used to construct  $\mathbf{v}^{**}$  and  $I^{**}$ :  $\mathbf{v}^{**}$  is the optimal estimate of the

motion field  $\mathbf{v}$  such that the corresponding  $I^{**}$  optimally fits the snapshots  $Y_s^{kl}$ .

As a result we define an iterative algorithm. Its  $i$ -iteration consists of the two following steps:

- 1) given the motion field estimate  $\hat{\mathbf{v}}^i = (\hat{u}^i, \hat{v}^i)^T$  we plug it into the optical flow constraint (2), that is:

$$\partial_t I = -\hat{u}^i(x, y, t)\partial_x I - \hat{v}^i(x, y, t)\partial_y I + e^o(x, y, t) \quad (4)$$

and compute the minimax estimate  $\widehat{I}^i$  of the brightness function  $I$  and estimate  $\widehat{\nabla I}^i$  of the image gradient  $\nabla I$ , considering (4) as a state equation and (1) as an observation equation. This is further detailed in 4.1;

- 2) given  $\widehat{\nabla I}^i$  and  $\hat{\mathbf{v}}^i$  we construct the following state equation:

$$\begin{aligned} \partial_t I &= -u(x, y, t)\widehat{\partial_x I}^i - v(x, y, t)\widehat{\partial_y I}^i + e^o(x, y, t), \\ \partial_t u + \hat{u}^i \partial_x u + \hat{v}^i \partial_y u + \partial_x p &= \nu \Delta u + e_u^m, \\ \partial_t v + \hat{u}^i \partial_x v + \hat{v}^i \partial_y v + \partial_y p &= \nu \Delta v + e_v^m, \\ \partial_x u + \partial_y v &= 0, \quad (x, y, t) \in \Omega_T, \\ \mathbf{v}(x, y, t) &= 0, \quad (x, y) \in \partial\Omega, \\ \mathbf{v}(x, y, t_0) &= \mathbf{v}_0(x, y) + \mathbf{e}^b(x, y), \quad (x, y) \in \Omega. \end{aligned} \quad (5)$$

and compute the minimax estimate  $\hat{\mathbf{v}}^{i+1}$  of the motion field considering the system of linear parabolic PDEs (5) as state equation and (1) as an observation equation.

#### 4.1 Minimax pseudo-observations

In this subsection, we construct the solution  $\hat{I}$  of (8) such that  $\hat{I}$  “fits” in the minimax sense the actually acquired data  $Y_s^{kl}$ , that is:

$$Y_s^{kl} = \int_{\Omega_T} g_s^{kl}(x, y, t) I(x, y, t) dx dy dt + \eta_s^{kl}, \quad (6)$$

where  $g_s^{kl} \in L^2(\Omega_T)$  reflects the discretization process linked to the acquisition procedure for the continuous images,  $\eta_s^{kl}$  is an observation noise representing the uncertainty introduced by acquisitions and discretization. We

assume that  $\eta_s^{kl}$  are realizations of independent random variables with zero mean and covariances  $R_s^{kl} = \mathbb{E}(\eta_s^{kl})^2$ . Define

$$H_s \varphi = \begin{bmatrix} \int_{t_0}^T \int_{\Omega} g_s^{11}(x, y, t) \varphi(x, y, t) dx dy dt \\ \vdots \\ \int_{t_0}^T \int_{\Omega} g_s^{MM}(x, y, t) \varphi(x, y, t) dx dy dt \end{bmatrix}$$

We stress that the introduced operators  $H_s$  are compact operators with finite dimensional range. This allows us to introduce the Moore-Penrose pseudo-inverse  $H_s^+$ , for each  $H_s$ , which is a linear bounded operator. We then set  $\bar{I}_0 := H_0^+ \{Y_0^{kl}\}_{k,l=1}^M$ .

Assume that  $\mathbf{v} = (u, v)^T$  with  $u, v \in C^0(t_0, T, L_{\infty}(\mathbb{R}^2))$  and let  $L_{\varepsilon}(t, \mathbf{v})$  denote a linear differential operator

$$L_{\varepsilon}(t, \mathbf{v}) := -u(t, x, y) \partial_x - v(t, x, y) \partial_y + \varepsilon^2 \Delta, \quad \varepsilon > 0, \quad (7)$$

where the diffusion term represents a regularization in the spirit of the vanishing viscosity method.

Let  $I$  denote the unique solution of the following linear evolution equation:

$$\begin{aligned} \frac{dI}{dt} &= L_{\varepsilon}(t, \mathbf{v})I + B e^o, \\ I(x, y, t_0) &= \bar{I}_0 + B_0 e^b(x, y), \quad N_{\Gamma} I = 0, \end{aligned} \quad (8)$$

with Neumann boundary condition, where  $\bar{I}_0 \in L^2(\Omega)$  represents initial condition, which is the initial image in our case,  $B_0$  and  $B$  are bounded linear operators on  $L^2(\Omega)$  that are supposed to introduce additional constraints on the uncertain parameters (for instance,  $B$  can be use to switch on or off the model error) and  $e^o, e^b$  are realizations of independent random processes, such that:

$$m_0(x, y) := \mathbb{E} e^b(x, y), \quad m_0 \in L^2(\Omega), \quad m(x, y, t) := \mathbb{E} e^o(x, y, t), \quad m \in L^2(\Omega_T) \quad (9)$$

and

$$\begin{aligned} Q_0(x, y, x', y') &:= \mathbb{E} e^b(x, y) e^b(x', y') \in L^2(\Omega \times \Omega), \\ Q(x, y, t, x', y', t') &:= \mathbb{E} e^o(x, y, t) e^o(x', y', t') \in L^2(\Omega_T \times \Omega_T). \end{aligned} \quad (10)$$

Let us note that (9) and (10) imply that realizations of  $e^b$  and  $e^o$  belong to  $L^2(\Omega)$  and  $L^2(\Omega_T)$  respectively. We also stress that the differential operator



$\frac{d}{dt} - L_\varepsilon(t, \mathbf{v})$ , with  $L_\varepsilon(t, \mathbf{v})$  defined by (7), is uniformly parabolic [2, p.369]. This allows to state that (see [2, p.352] for details) (8) has a unique weak solution  $I \in \mathbb{L}^2(t_0, T, H^1(\Omega)) \cap \mathbb{C}(t_0, T, L^2(\Omega))$  for each realization of  $e^b$  and  $e^o$ .

**Proposition 1** *The linear minimax estimate  $\widehat{I}$  of  $I$  solves the following system of equations:*

$$\begin{aligned} \frac{d\widehat{I}}{dt} &= L_\varepsilon(t, \mathbf{v})\widehat{I} + BQ^{-1}B^*\widehat{p}, \\ \widehat{I}(x, y, t_0) &= \bar{I}_0 + B_0Q_0B_0^*\widehat{p}(x, y, t_0), \quad N_\Gamma\widehat{I} = 0, \\ -\frac{d\widehat{p}}{dt} &= L_\varepsilon^*(t, \mathbf{v})\widehat{p} + \sum_{s=1}^N \sum_{k,l=1}^M g_s^{kl}(R_s^{kl})^{-1}(Y_s^{kl} - \int_{\Omega_T} g_s^{kl}\widehat{I} dx dy dt), \\ \widehat{p}(x, y, T) &= 0, \quad N_\Gamma\widehat{p} = 0. \end{aligned} \tag{11}$$

We note that, by assumption, see (3), we have  $\mathbf{v}(x, y, t) = 0$  for  $(x, y) \in \partial\Omega$ . The Gauss-Ostrogradsky theorem provides:

$$L_\varepsilon^*(t, \mathbf{v}) = u(x, y, t)\partial_x + v(x, y, t)\partial_y + \varepsilon^2\Delta$$

In what follows, we transform the boundary value problem (11) into a number of independent linear problems. To do so, we introduce functions  $\bar{I}$ ,  $I_s^{kl}$  and  $p_s^{kl}$ , defined on  $\Omega_T$  as solutions of the following equations:

$$\begin{aligned} \partial_t \bar{I} &= L_\varepsilon(t, \mathbf{v})\bar{I}, \quad \bar{I}(x, y, t_0) = \bar{I}_0, \quad N_\Gamma\bar{I} = 0, \\ \partial_t I_s^{kl} &= L_\varepsilon(t, \mathbf{v})I_s^{kl} + BQ^{-1}B^*p_s^{kl}, \\ I_s^{kl}(x, y, t_0) &= B_0Q_0B_0^*p_s^{kl}(x, y, t_0), \quad N_\Gamma I_s^{kl} = 0, \\ -\partial_t p_s^{kl} &= L_\varepsilon^*(t, \mathbf{v})p_s^{kl} + g_s^{kl}, \quad p_s^{kl}(x, y, T) = 0, \quad N_\Gamma p_s^{kl} = 0. \end{aligned} \tag{12}$$

We assume that the following representation holds for some  $\beta_s^{kl}$ :

$$\widehat{I} = \bar{I} + \sum_{s=1}^N \sum_{k,l=1}^M (R_s^{kl})^{-1}(Y_s^{kl} - \beta_s^{kl})I_s^{kl}, \tag{13}$$

and

$$\widehat{p} = \sum_{s=1}^N \sum_{k,l=1}^M (R_s^{kl})^{-1}(Y_s^{kl} - \beta_s^{kl})p_s^{kl}. \tag{14}$$

In order to find coefficients  $\beta_s^{kl}$ , we substitute (14) into (11). We get, using the last equation in (12), that

$$-\frac{d\hat{p}}{dt} = L_\varepsilon^*(t, \mathbf{v})\hat{p} + \sum_{s=1}^N \sum_{k,l=1}^M (R_s^{kl})^{-1} (Y_s^{kl} - \beta_s^{kl}) g_s^{kl}.$$

Equating the coefficient in front of  $g_s^{kl}$  in the obtained formula with the corresponding coefficient in (11), we get the following system of linear algebraic equations for determining<sup>1</sup>  $\beta_s^{kl}$ :

$$\begin{aligned} (R_s^{kl})^{-1} (Y_s^{kl} - \beta_s^{kl}) &= (R_s^{kl})^{-1} (Y_s^{kl} - \int_{\Omega_T} g_s^{kl} \widehat{I} dx dy dt) \\ &= (R_s^{kl})^{-1} (Y_s^{kl} - \int_{\Omega_T} g_s^{kl} \bar{I} dx dy dt) \\ &\quad - (R_s^{kl})^{-1} \left( \sum_{s'=1}^N \sum_{k',l'=1}^M (R_{s'}^{k'l'})^{-1} (Y_{s'}^{k'l'} - \beta_{s'}^{k'l'}) \int_{\Omega_T} g_s^{kl} I_{s'}^{k'l'} dx dy dt \right) \end{aligned}$$

We then get the following system ( $k = \overline{1, N_x}, l = \overline{1, N_y}, s = \overline{1, S}$ ):

$$\begin{aligned} \beta_s^{kl} + \sum_{s'=1}^N \sum_{k',l'=1}^M \left[ \int_{\Omega_T} g_s^{kl} (R_{s'}^{k'l'})^{-1} I_{s'}^{k'l'} dx dy dt \right] \beta_{s'}^{k'l'} \\ = \int_{\Omega_T} g_s^{kl} \bar{I} dx dy dt + \sum_{s'=1}^N \sum_{k',l'=1}^M \left[ \int_{\Omega_T} g_s^{kl} (R_{s'}^{k'l'})^{-1} I_{s'}^{k'l'} dx dy dt \right] Y_{s'}^{k'l'}. \end{aligned} \tag{15}$$

Let us define:

$$\mathbf{A} := \begin{bmatrix} \frac{\langle g_1^{11}, I_1^{11} \rangle}{R_1^{11}} & \dots & \frac{\langle g_1^{11}, I_1^{N_x 1} \rangle}{R_1^{N_x 1}} & \dots & \frac{\langle g_1^{11}, I_1^{N_x N_y} \rangle}{R_1^{N_x N_y}} & \dots & \frac{\langle g_1^{11}, I_{N_s}^{N_x N_y} \rangle}{R_{N_s}^{N_x N_y}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\langle g_1^{N_x 1}, I_1^{11} \rangle}{R_1^{11}} & \dots & \frac{\langle g_1^{N_x 1}, I_1^{N_x 1} \rangle}{R_1^{N_x 1}} & \dots & \frac{\langle g_1^{N_x 1}, I_1^{N_x N_y} \rangle}{R_1^{N_x N_y}} & \dots & \frac{\langle g_1^{N_x 1}, I_{N_s}^{N_x N_y} \rangle}{R_{N_s}^{N_x N_y}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\langle g_1^{N_x N_y}, I_1^{11} \rangle}{R_1^{11}} & \dots & \frac{\langle g_1^{N_x N_y}, I_1^{N_x 1} \rangle}{R_1^{N_x 1}} & \dots & \frac{\langle g_1^{N_x N_y}, I_1^{N_x N_y} \rangle}{R_1^{N_x N_y}} & \dots & \frac{\langle g_1^{N_x N_y}, I_{N_s}^{N_x N_y} \rangle}{R_{N_s}^{N_x N_y}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\langle g_{N_s}^{N_x N_y}, I_1^{11} \rangle}{R_1^{11}} & \dots & \frac{\langle g_{N_s}^{N_x N_y}, I_1^{N_x 1} \rangle}{R_1^{N_x 1}} & \dots & \frac{\langle g_{N_s}^{N_x N_y}, I_1^{N_x N_y} \rangle}{R_1^{N_x N_y}} & \dots & \frac{\langle g_{N_s}^{N_x N_y}, I_{N_s}^{N_x N_y} \rangle}{R_{N_s}^{N_x N_y}} \end{bmatrix}$$

<sup>1</sup>(13) is applied to pass from the first line to the second line.

and set:

$$\boldsymbol{\beta} := (\beta_1^{11} \dots \beta_1^{N_x 1} \dots \beta_1^{N_x N_y} \dots \beta_{N_s}^{N_x N_y})^T ,$$

and:

$$\mathbf{v} := (\langle g_1^{11}, \bar{I} \rangle \dots \langle g_1^{N_x 1}, \bar{I} \rangle \dots \langle g_1^{N_x N_y}, \bar{I} \rangle \dots \langle g_{N_s}^{N_x N_y}, \bar{I} \rangle)^T .$$

We find that (15) is equal to the following system of linear algebraic equations:

$$(\mathbf{I} + \mathbf{A})\boldsymbol{\beta} = \mathbf{v} + \mathbf{A}\mathbf{Y} .$$

If (15) has a solution then, by construction  $\widehat{I}$  and  $\hat{p}$  defined by (13)-(14), with  $\beta_s^{kl}$  determined from (15), solve (11). On the other hand, as (11) is uniquely solvable, it follows that the solution of (11) coincides with that of (13)-(14). We stress that the system (11) has a solution, by proposition 1, and, therefore, it necessarily has the form (13)-(14). This proves that there is at least one vector  $\boldsymbol{\beta}$  solving the system (15).

As a result, we get  $\widehat{I}$ , which represents the minimax estimate of the brightness function and allows to get an optimal minimax estimate of the image gradient  $\widehat{\nabla I^i}$  - one basically uses gradients of  $\bar{I}$  and  $I_s^{kl}$  which has been computed during the integration of (12).

## 4.2 Minimax optical flow estimation

Let  $\widehat{\nabla I^i}$  denote the minimax estimate of the image gradient  $\nabla I$  and  $\hat{\mathbf{v}}^i = (\hat{u}^i, \hat{v}^i)^T$  stands for the linear minimax estimate of  $\mathbf{v}$ . Define the advection-diffusion operator

$$A_\varepsilon(t, \hat{\mathbf{v}}^i)\mathbf{v} = \begin{bmatrix} L_\varepsilon(t, \hat{\mathbf{v}}^i)u \\ L_\varepsilon(t, \hat{\mathbf{v}}^i)v \end{bmatrix} .$$

Let us recall that

$$L_\varepsilon(t, \mathbf{v}) := -u(t, x, y)\partial_x - v(t, x, y)\partial_y + \varepsilon^2 \Delta, \quad \varepsilon > 0 ,$$

and

$$L_\varepsilon^*(t, \mathbf{v}) = u(x, y, t)\partial_x + v(x, y, t)\partial_y + \varepsilon^2 \Delta .$$

Given observations in the form:

$$Y_s^{kl} = \int_{\Omega_T} g_s^{kl}(x, y, t) I(x, y, t) dx dy dt + \eta_s^{kl} , \quad (16)$$

let us construct the minimax estimate of the state of the following PDE:

$$\begin{aligned}
\frac{dI}{dt} &= L_\varepsilon(t, \mathbf{v})\widehat{I}^i + B_I e^o(x, y, t), \\
\partial_t \mathbf{v} &= A_\varepsilon(t, \widehat{\mathbf{v}}^i) \mathbf{v} - \nabla p + B_v \mathbf{e}^m, \\
\operatorname{div} \mathbf{v} &= 0, \quad (t, x, y) \in \Omega_T, \\
\mathbf{v}(x, y, t) &= 0, \quad (x, y) \in \partial\Omega, \\
\mathbf{v}(x, y, t_0) &= \mathbf{v}_0(x, y) + \mathbf{e}^b(x, y), \quad (x, y) \in \Omega.
\end{aligned} \tag{17}$$

Next proposition describes the minimax estimates of the velocity field  $\mathbf{v}$  and brightness function  $I$ .

**Proposition 2** *The linear minimax estimates  $\widehat{\mathbf{v}}, \widehat{I}$  of  $\mathbf{v}, I$  solve the following system of equations:*

$$\begin{aligned}
\partial_t \mathbf{v} &= A_\varepsilon(t, \mathbf{v}^i) \mathbf{v} - \nabla p + B_v Q_v^{-1} B_v^* \mathbf{q}, \\
\partial_t \mathbf{q} &= -A_\varepsilon^*(t, \mathbf{v}^i) \mathbf{q} + \nabla w + \mu(x, y, t) \nabla \widehat{I}^i, \\
-\partial_t \mu &= \sum_{s=1}^N \sum_{k,l=1}^M g_s^{kl} (R_s^{kl})^{-1} (Y_s^{kl} - \int_{\Omega_T} g_s^{kl} \widehat{I} dx dy dt), \\
\partial_t \widehat{I} &= L_\varepsilon(t, \mathbf{v}) \widehat{I}^i + B_I Q_I^{-1} B_I^* \mu, \\
\operatorname{div} \mathbf{v} &= \operatorname{div} \mathbf{q} = 0, \quad (t, x, y) \in \Omega_T, \\
\mathbf{v}(x, y, t) &= \mathbf{q}(x, y, t) = 0, \quad (x, y) \in \partial\Omega, \\
\mathbf{v}(x, y, t_0) &= \overline{\mathbf{v}}_0 + B_0^v Q_0^v (B_0^v)^* \mathbf{q}(x, y, t_0), \\
\mathbf{q}(x, y, T) &= 0, \quad \mu(x, y, T) = 0, \quad N_\Gamma \mu = 0, \\
\widehat{I}(x, y, t_0) &= \overline{I}_0 + B_0^I Q_0^I (B_0^I)^* \mu(x, y, t_0), \quad N_\Gamma \widehat{I} = 0,
\end{aligned} \tag{18}$$

Using the same idea as above, we introduce the following functions:

$$\begin{aligned}
\partial_t \mathbf{v}_s^{k,l} &= A_\varepsilon(t, \mathbf{v}^i) \mathbf{v}_s^{k,l} - \nabla p_s^{k,l} + B_v Q_v^{-1} B_v^* \mathbf{q}_s^{k,l}, \\
\partial_t \mathbf{q}_s^{k,l} &= -A_\varepsilon^*(t, \mathbf{v}^i) \mathbf{q}_s^{k,l} + \nabla w_s^{k,l} + \mu_s^{k,l} \nabla \widehat{I}^i, \\
-\partial_t \mu_s^{k,l} &= g_s^{kl}, \\
\partial_t \widehat{I} &= L_\varepsilon(t, \mathbf{v}_s^{k,l}) \widehat{I}^i + B_I Q_I^{-1} B_I^* \mu_s^{k,l}, \\
\operatorname{div} \mathbf{v}_s^{k,l} &= \operatorname{div} \mathbf{q}_s^{k,l} = 0, \quad (t, x, y) \in \Omega_T, \\
\mathbf{v}_s^{k,l}(x, y, t) &= \mathbf{q}_s^{k,l}(x, y, t) = 0, \quad (x, y) \in \partial\Omega, \\
\mathbf{v}_s^{k,l}(x, y, t_0) &= B_0^v Q_0^v (B_0^v)^* \mathbf{q}_s^{k,l}(x, y, t_0), \\
\mathbf{q}_s^{k,l}(x, y, T) &= 0, \quad \mu_s^{k,l}(x, y, T) = 0, \quad N_\Gamma \mu = 0, \\
I_s^{k,l}(x, y, t_0) &= B_0^I Q_0^I (B_0^I)^* \mu_s^{k,l}(x, y, t_0), \quad N_\Gamma \widehat{I} = 0,
\end{aligned} \tag{19}$$

Now, we note that there are coefficients  $\beta_s^{kl}$  such that:

$$\hat{I} = \bar{I} + \sum_{s=1}^N \sum_{k,l=1}^M (R_s^{kl})^{-1} (Y_s^{kl} - \beta_s^{kl}) I_s^{kl}, \quad (20)$$

and

$$\hat{\mathbf{v}} = \sum_{s=1}^N \sum_{k,l=1}^M (R_s^{kl})^{-1} (Y_s^{kl} - \beta_s^{kl}) p_s^{kl}. \quad (21)$$

where  $\beta_s^{kl}$  solve a system of linear algebraic equations similar to (15).

## 5 Conclusion

This paper presents an iterative minimax state estimation algorithm. It aims to: first, optimally resolve the issue of sparse temporal acquisitions; second, avoid modelling the uncertainty parameters by Gaussian variables; and last provide an uncertainty measure associated to the estimation. Further experiments are needed to assess its quality on real satellite images.

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